N-Phase Elliptical Inhomogeneities With Internal Uniform Stresses in Plane Elasticity

Xu Wang
School of Mechanical Engineering,
Zhengzhou University,
Zhengzhou,
Henan 450001, P. R. China;
Center for Composite Materials,
University of Delaware,
Newark, DE 19716

We investigate the problem of an N-phase elliptical inhomogeneity in plane elasticity. The elliptical inhomogeneity is bonded to the unbounded matrix through the intermediate $(N-2)$ interphases, and the matrix is subjected to remote uniform stresses. We observe that the stress field inside the elliptical inhomogeneity is still uniform when the following two conditions are satisfied: (i) The formed interfaces are $(N-1)$ confocal ellipses, and (ii) the interphases and the matrix possess the same shear modulus but different Poisson's ratios. In Appendixes A and B, we also discuss an arbitrary number of interacting arbitrary shaped inhomogeneities embedded in an infinite matrix, and an N-phase inhomogeneity with $(N-1)$ interfaces of arbitrary shape. Here all the phases comprising the composite possess the same shear modulus but different Poisson's ratios. The results in the main body and in Appendixes A and B are further extended in Appendix C to finite plane strain deformations of compressible hyperelastic harmonic materials.
[DOI: 10.1115/1.4000929]

Keywords: layered elliptical inhomogeneity, uniform stress field, arbitrary shaped inhomogeneity, singularity, analytic continuation

1 Introduction

It has been found that the stress field inside an N-phase elliptical inhomogeneity is uniform when the formed interfaces are confocal ellipses and when the matrix is subjected to remote uniform antiplane shear stresses [1]. This conclusion, however, is usually not valid when the same layered structure is subjected to remote uniform in-plane stresses. For example, even the stress field within a three-phase concentric circular inhomogeneity is intrinsically nonuniform when the matrix is subjected to remote uniform nonhydrostatic in-plane stresses [2,3]. Only when the relative thickness of the interphase layer and the aspect ratio of the elliptical inhomogeneity satisfy a relationship for given remote stresses will the internal stress field of a three-phase inhomogeneity be uniform and hydrostatic [4]. The main purpose of this research is to probe the existence of a uniform (but not necessarily hydrostatic) stress field within an N-phase elliptical inhomogeneity under in-plane deformations in the absence of the thickness and aspect ratio relationship in Ref. [4].

On the other hand, when discussing the problem of an isolated arbitrary shaped inhomogeneity, it is necessary to use the Faber polynomials to calculate the elastic field in the composite [5]. Furthermore it seems intractable to analytically solve the problem of interacting arbitrary shaped inhomogeneities. In Appendix A, we find that strikingly simple solutions still exist for interacting arbitrary shaped inhomogeneities when the inhomogeneities and the surrounding matrix possess the same shear modulus but different Poisson's ratios. Particularly it is observed that the stress field inside an isolated inhomogeneity can be conveniently determined. As an extension, in Appendix B, we address an N-phase inhomogeneity with $(N-1)$ interfaces of arbitrary shape. Here the composite possesses a constant shear modulus. It is observed that (i) the internal stresses are still uniform when all the interfaces are ellipses (here the elliptical interfaces are not confined to confocal and the centers of the ellipses are not necessarily common), and (ii) the uniformity of the internal stresses for a multiphase inhomogeneity with nonelliptical interfaces is possible. Finally, in Appendix C, we discuss a composite made up of compressible hyperelastic harmonic materials.

Contributed by the Applied Mechanics of ASME for publication in the Journal of Applied Mechanics. Manuscript received September 16, 2009; final manuscript received November 29, 2009; published online April 19, 2010. Assoc. Editor: Pradeep Sharma.

2 Formulation

For plane deformations of an isotropic elastic material, the in-plane displacements u and v, the two resultant forces F_x and F_y, and the in-plane stresses σ_{xx}, σ_{yy}, and σ_{xy} can be expressed in terms of two analytic functions $\phi(z)$ and $\psi(z)$ of the complex variable $z=x+iy$ as [6]

$$2\mu(u+iv)=\kappa\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}$$

$$F_x+iF_y=-i[\phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)}] \tag{1}$$

$$\sigma_{xx}+\sigma_{yy}=2[\phi'(z)+\overline{\phi'(z)}]$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy}=2[\bar{z}\phi''(z)+\psi'(z)] \tag{2}$$

where $\kappa=3-4\nu$ for plane strain, which is assumed in this investigation, and $\kappa=(3-\nu)/(1+\nu)$ for plane stress; μ and ν, where $\mu>0$ and $0<\nu\leq 0.5$, are the shear modulus and Poisson's ratio, respectively.

We consider an N-phase elliptical inhomogeneity with $(N-1)$ confocal interfaces. Let S_1, S_m $[m=2\sim(N-1)]$, and S_N denote the inhomogeneity, the $(N-2)$ interphases, and the unbounded matrix, all of which are perfectly bonded through the $(N-1)$ confocal interfaces Γ_m $[m=1\sim(N-1)]$. The phase numbers are sequentially ordered from 1 for the innermost inhomogeneity to N for the outermost matrix, and the interface Γ_m is formed by S_m and S_{m+1}. The inhomogeneity is centered at the origin and its major and minor axes are along the x and y axes, respectively. In the following discussions the subscripts 1, m $[m=2\sim(N-1)]$, and N are used to identify the associated quantities in S_1, S_m $[m=2\sim(N-1)]$, and S_N. In this work we confine our attention to the case in which the $(N-2)$ interphases and the matrix possess the same shear modulus; i.e., $\mu_m=\mu_N$ $[m=2\sim(N-1)]$. Here we point out that the assumption of equal shear modulus has been adopted in literature to arrive at analytical solutions (see, for example, Refs. [7,8]). The common foci of the $(N-1)$ confocal interfaces are at $x=\pm c$. Now we consider the following conformal mapping:

$$z=\omega(\zeta)=\frac{c}{2}(\zeta+\zeta^{-1}) \tag{3}$$

which maps the segment $[-c,\ c]$ onto the unit circle in the ζ-plane, and the interfaces Γ_m $[m=1\sim(N-1)]$ onto $(N-1)$ coaxial circles with radii R_m $[m=1\sim(N-1)]$. For convenience we write $\phi_i(\zeta)=\phi_i(\omega(\zeta))$, $\psi_i(\zeta)=\psi_i(\omega(\zeta))$ $(i=1\sim N)$.

3 The Internal Uniform Stress Field

There are in total $(N-1)$ interfaces Γ_m $[m=1\sim(N-1)]$ in the composite. Among these interfaces, the $(N-2)$ interfaces Γ_m $[m=2\sim(N-1)]$ are simpler to treat because the adjacent phases forming any one of the $(N-2)$ interfaces possess the same shear modulus. The continuity conditions of tractions and displacements across the interfaces $|\zeta|=R_m$ $[m=2\sim(N-1)]$ can be expressed as

$$\phi_m(\zeta)+\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi_m'(\zeta)}+\overline{\psi_m(\zeta)}=\phi_{m+1}(\zeta)+\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi_{m+1}'(\zeta)}+\overline{\psi_{m+1}(\zeta)}$$

$$\kappa_m\phi_m(\zeta)-\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi_m'(\zeta)}-\overline{\psi_m(\zeta)}=\kappa_{m+1}\phi_{m+1}(\zeta)-\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi_{m+1}'(\zeta)}-\overline{\psi_{m+1}(\zeta)}$$

$$|\zeta|=R_m\quad[m=2\sim(N-1)] \tag{4}$$

Adding the two expressions in Eq. (4) together, we arrive at

$$(\kappa_m+1)\phi_m(\zeta)=(\kappa_{m+1}+1)\phi_{m+1}(\zeta),\quad |\zeta|=R_m\quad[m=2\sim(N-1)] \tag{5}$$

We then introduce an analytic function $\phi_0(\zeta)$ defined by

$$\phi_0(\zeta)=\frac{\kappa_m+1}{\kappa_N+1}\phi_m(\zeta)\quad(m=2\sim N) \tag{6}$$

Apparently $\phi_0(\zeta)$ is continuous across the interfaces $|\zeta|=R_m$ $[m=2\sim(N-1)]$ and then analytic within the region $|\zeta|>R_1$ except at infinity. The singular behavior of $\phi_0(\zeta)$ at infinity is

$$\phi_0(\zeta)=A\zeta+o(1),\quad |\zeta|\to\infty \tag{7}$$

where the real constant A is related to the remote uniform stresses through $A=c(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty})/8$. By using Eq. (6), then the interface conditions in Eq. (4) can be re-expressed into

$$\psi_m(\zeta)+\left(\frac{\kappa_N+1}{\kappa_m+1}-\frac{\kappa_N+1}{\kappa_{m+1}+1}\right)\bar{\phi}_0(R_m^2/\zeta)$$
$$=\psi_{m+1}(\zeta)+\left(\frac{\kappa_N+1}{\kappa_{m+1}+1}-\frac{\kappa_N+1}{\kappa_m+1}\right)\frac{\bar{\omega}(R_m^2/\zeta)}{\omega'(\zeta)}\phi_0'(\zeta)$$
$$|\zeta|=R_m\quad[m=2\sim(N-1)] \tag{8}$$

In view of Eq. (8), we then introduce another analytic function $\psi_0(\zeta)$ defined by

$$\psi_0(\zeta)=\psi_2(\zeta)+\sum_{j=2}^{N-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\bar{\phi}_0(R_j^2/\zeta)$$
$$=\psi_m(\zeta)+\sum_{j=m}^{N-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\bar{\phi}_0(R_j^2/\zeta)$$
$$-\sum_{j=2}^{m-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\frac{\bar{\omega}(R_j^2/\zeta)}{\omega'(\zeta)}\phi_0'(\zeta)$$
$$[m=3\sim(N-1)]$$
$$=\psi_N(\zeta)-\sum_{j=2}^{N-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\frac{\bar{\omega}(R_j^2/\zeta)}{\omega'(\zeta)}\phi_0'(\zeta) \tag{9}$$

It can be easily observed that $\psi_0(\zeta)$ is also continuous across the interfaces $|\zeta|=R_m$ $[m=2\sim(N-1)]$ and then analytic within the region $|\zeta|>R_1$ except at infinity. The singular behavior of $\psi_0(\zeta)$ at infinity is

$$\psi_0(\zeta)=\left[B-A\sum_{j=2}^{N-1}R_j^{-2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\right]\zeta+o(1),\quad |\zeta|\to\infty \tag{10}$$

where the complex constant B is related to the remote uniform stresses through $B=c(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty}+2i\sigma_{xy}^{\infty})/4$.

To ensure the uniform stress field inside the elliptical inhomogeneity, we have to assume that $\phi_1(\zeta)$ and $\psi_1(\zeta)$ must take the following forms:

$$\phi_1(\zeta)=D(\zeta+\zeta^{-1}),\quad \psi_1(\zeta)=E(\zeta+\zeta^{-1}) \tag{11}$$

where D and E are two constants to be determined. Our task below is to determine D and E. The continuity conditions of tractions and displacements across the inner interface $|\zeta|=R_1$ can be expressed into

$$\phi_2(\zeta)+\frac{\omega(\zeta)}{\bar{\omega}'(R_1^2/\zeta)}\bar{\phi}_2'(R_1^2/\zeta)+\bar{\psi}_2(R_1^2/\zeta)$$
$$=(D+\bar{D}+R_1^{-2}\bar{E})\zeta+(D+\bar{D}+R_1^2\bar{E})\zeta^{-1}$$

$$\kappa_2\phi_2(\zeta)-\frac{\omega(\zeta)}{\bar{\omega}'(R_1^2/\zeta)}\bar{\phi}_2'(R_1^2/\zeta)-\bar{\psi}_2(R_1^2/\zeta)$$
$$=\Gamma(\kappa_1D-\bar{D}-R_1^{-2}\bar{E})\zeta+\Gamma(\kappa_1D-\bar{D}-R_1^2\bar{E})\zeta^{-1},\quad |\zeta|=R_1 \tag{12}$$

where $\Gamma=\mu_N/\mu_1$. The above can be equivalently expressed in terms of $\phi_0(\zeta)$ and $\psi_0(\zeta)$ as

$$\frac{\kappa_N+1}{\kappa_2+1}\phi_0(\zeta)-\sum_{j=2}^{N-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\phi_0\left(\frac{R_j^2}{R_1^2}\zeta\right)-A\left[\frac{\kappa_N+1}{\kappa_2+1}\right.$$
$$\left.-\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\right]\zeta+\left[R_1^2\bar{B}+A\frac{\kappa_N+1}{\kappa_2+1}\right.$$
$$\left.-AR_1^2\sum_{j=2}^{N-1}R_j^{-2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)-(D+\bar{D}+R_1^2\bar{E})\right]\zeta^{-1}$$
$$=-\bar{\psi}_0(R_1^2/\zeta)-\frac{\kappa_N+1}{\kappa_2+1}\frac{\omega(\zeta)}{\bar{\omega}'(R_1^2/\zeta)}\bar{\phi}_0'(R_1^2/\zeta)-\left[A\left[\frac{\kappa_N+1}{\kappa_2+1}\right.\right.$$
$$\left.\left.-\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\right]-(D+\bar{D}+R_1^{-2}\bar{E})\right]\zeta+\left[R_1^2\bar{B}\right.$$

$$+A\frac{\kappa_N+1}{\kappa_2+1}-AR_1^2\sum_{j=2}^{N-1}R_j^{-2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\Bigg]\zeta^{-1},\quad |\zeta|=R_1 \tag{13}$$

$$\frac{\kappa_2(\kappa_N+1)}{\kappa_2+1}\phi_0(\zeta)+\sum_{j=2}^{N-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\phi_0\left(\frac{R_j^2}{R_1^2}\zeta\right)$$
$$-A\left[\frac{\kappa_2(\kappa_N+1)}{\kappa_2+1}+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\right]\zeta-\Bigg[R_1^2\bar{B}$$
$$+A\frac{\kappa_N+1}{\kappa_2+1}-AR_1^2\sum_{j=2}^{N-1}R_j^{-2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)+\Gamma(\kappa_1 D-\bar{D}$$
$$-R_1^2\bar{E})\Bigg]\zeta^{-1}=\bar{\psi}_0(R_1^2/\zeta)+\frac{\kappa_N+1}{\kappa_2+1}\frac{\omega(\zeta)}{\bar{\omega}'(R_1^2/\zeta)}\bar{\phi}_0'(R_1^2/\zeta)$$
$$-\Bigg[A\left[\frac{\kappa_2(\kappa_N+1)}{\kappa_2+1}+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\right]-\Gamma(\kappa_1 D$$
$$-\bar{D}-R_1^{-2}\bar{E})\Bigg]\zeta-\Bigg[R_1^2\bar{B}+A\frac{\kappa_N+1}{\kappa_2+1}-AR_1^2\sum_{j=2}^{N-1}R_j^{-2}\left(\frac{\kappa_N+1}{\kappa_j+1}\right.$$
$$\left.-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\Bigg]\zeta^{-1},\quad |\zeta|=R_1 \tag{14}$$

Apparently the left-hand side of Eq. (13) and that of Eq. (14) are analytic outside the circle $|\zeta|=R_1$ including the point at infinity, while the right-hand side of Eq. (13) and that of Eq. (14) are analytic within the circle $|\zeta|=R_1$. Consequently the two sides of both Eqs. (13) and (14) should be identically zero such that

$$\frac{\kappa_N+1}{\kappa_2+1}\phi_0(\zeta)-\sum_{j=2}^{N-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\phi_0\left(\frac{R_j^2}{R_1^2}\zeta\right)=A\Bigg[\frac{\kappa_N+1}{\kappa_2+1}$$
$$-\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\Bigg]\zeta-\Bigg[R_1^2\bar{B}+A\frac{\kappa_N+1}{\kappa_2+1}$$
$$-A\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)-(D+\bar{D}+R_1^2\bar{E})\Bigg]\zeta^{-1} \tag{15}$$

$$\bar{\psi}_0(R_1^2/\zeta)+\frac{\kappa_N+1}{\kappa_2+1}\frac{\omega(\zeta)}{\bar{\omega}'(R_1^2/\zeta)}\bar{\phi}_0'(R_1^2/\zeta)=-\Bigg\{A\Bigg[\frac{\kappa_N+1}{\kappa_2+1}$$
$$-\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\Bigg]-(D+\bar{D}+R_1^{-2}\bar{E})\Bigg\}\zeta+\Bigg[R_1^2\bar{B}$$
$$+A\frac{\kappa_N+1}{\kappa_2+1}-A\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\Bigg]\zeta^{-1} \tag{16}$$

$$\frac{\kappa_2(\kappa_N+1)}{\kappa_2+1}\phi_0(\zeta)+\sum_{j=2}^{N-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\phi_0\left(\frac{R_j^2}{R_1^2}\zeta\right)$$
$$=A\left[\frac{\kappa_2(\kappa_N+1)}{\kappa_2+1}+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\right]\zeta+\Bigg[R_1^2\bar{B}$$
$$+A\frac{\kappa_N+1}{\kappa_2+1}-A\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)$$
$$+\Gamma(\kappa_1 D-\bar{D}-R_1^2\bar{E})\Bigg]\zeta^{-1} \tag{17}$$

$$\bar{\psi}_0(R_1^2/\zeta)+\frac{\kappa_N+1}{\kappa_2+1}\frac{\omega(\zeta)}{\bar{\omega}'(R_1^2/\zeta)}\bar{\phi}_0'(R_1^2/\zeta)=\Bigg[A\Bigg[\frac{\kappa_2(\kappa_N+1)}{\kappa_2+1}$$
$$+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\Bigg]-\Gamma(\kappa_1 D-\bar{D}-R_1^{-2}\bar{E})\Bigg]\zeta$$
$$+\left[R_1^2\bar{B}+A\frac{\kappa_N+1}{\kappa_2+1}-A\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\right]\zeta^{-1} \tag{18}$$

The solutions of $\phi_0(\zeta)$ can be easily obtained from Eqs. (15) and (17) as

$$\phi_0(\zeta)=A\zeta-\frac{R_1^2\bar{B}+A\frac{\kappa_N+1}{\kappa_2+1}-A\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)-(D+\bar{D}+R_1^2\bar{E})}{\frac{\kappa_N+1}{\kappa_2+1}-\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)}\zeta^{-1} \tag{19}$$

$$\phi_0(\zeta)=A\zeta+\frac{R_1^2\bar{B}+A\frac{\kappa_N+1}{\kappa_2+1}-A\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)+\Gamma(\kappa_1 D-\bar{D}-R_1^2\bar{E})}{\frac{\kappa_2(\kappa_N+1)}{\kappa_2+1}+\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)}\zeta^{-1} \tag{20}$$

In fact, the above two expressions of $\phi_0(\zeta)$ must be exactly the same, then we arrive at the following relationship between the two constants D and E:

$$-\frac{R_1^2\bar{B}+A\frac{\kappa_N+1}{\kappa_2+1}-A\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)-(D+\bar{D}+R_1^2\bar{E})}{\frac{\kappa_N+1}{\kappa_2+1}-\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)}=\frac{R_1^2\bar{B}+A\frac{\kappa_N+1}{\kappa_2+1}-A\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)+\Gamma(\kappa_1 D-\bar{D}-R_1^2\bar{E})}{\frac{\kappa_2(\kappa_N+1)}{\kappa_2+1}+\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)} \tag{21}$$

In addition the two expressions of $\bar{\psi}_0(R_1^2/\zeta)+((\kappa_N+1)/(\kappa_2+1))(\omega(\zeta)/(\bar{\omega}'(R_1^2/\zeta)))\bar{\phi}_0'(R_1^2/\zeta)$ obtained in Eqs. (16) and (18) also must be exactly the same; we then obtain the following simple relationship between D and E:

$$E=\frac{R_1^2(1+\Gamma\kappa_1)}{\Gamma-1}\bar{D}-R_1^2 D+\frac{AR_1^2(\kappa_N+1)}{1-\Gamma} \tag{22}$$

The two constants D and E can then be uniquely obtained from Eqs. (21) and (22) as follows:

$$\mathrm{Re}\{D\}=\frac{R_1^2(\Gamma-1)\mathrm{Re}\{B\}+A(\kappa_N+1)\left[R_1^4+(\Gamma-1)(R_1^4+1)\left[\frac{1}{\kappa_1+1}-\sum_{j=1}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{1}{\kappa_j+1}-\frac{1}{\kappa_{j+1}+1}\right)\right]\right]}{2(\Gamma-1)+R_1^4(2+\Gamma\kappa_1-\Gamma)+(\Gamma-1)(2+\Gamma\kappa_1-\Gamma)(R_1^4-1)\left[\frac{1}{\kappa_1+1}-\sum_{j=1}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{1}{\kappa_j+1}-\frac{1}{\kappa_{j+1}+1}\right)\right]}$$

$$\mathrm{Im}\{D\}=\frac{R_1^2(1-\Gamma)\mathrm{Im}\{B\}}{\Gamma(\kappa_1+1)\left[R_1^4+(\Gamma-1)(R_1^4-1)\left[\frac{1}{\kappa_1+1}-\sum_{j=1}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{1}{\kappa_j+1}-\frac{1}{\kappa_{j+1}+1}\right)\right]\right]} \tag{23}$$

$$\mathrm{Re}\{E\}=\frac{R_1^4(2+\Gamma\kappa_1-\Gamma)\mathrm{Re}\{B\}-2R_1^2A(\kappa_N+1)\left[1-(2+\Gamma\kappa_1-\Gamma)\left[\frac{1}{\kappa_1+1}-\sum_{j=1}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{1}{\kappa_j+1}-\frac{1}{\kappa_{j+1}+1}\right)\right]\right]}{2(\Gamma-1)+R_1^4(2+\Gamma\kappa_1-\Gamma)+(\Gamma-1)(2+\Gamma\kappa_1-\Gamma)(R_1^4-1)\left[\frac{1}{\kappa_1+1}-\sum_{j=1}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{1}{\kappa_j+1}-\frac{1}{\kappa_{j+1}+1}\right)\right]}$$

$$\mathrm{Im}\{E\}=\frac{R_1^4\,\mathrm{Im}\{B\}}{R_1^4+(\Gamma-1)(R_1^4-1)\left[\frac{1}{\kappa_1+1}-\sum_{j=1}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{1}{\kappa_j+1}-\frac{1}{\kappa_{j+1}+1}\right)\right]} \tag{24}$$

The uniform stress field within the elliptical inhomogeneity is given by

$$\sigma_{xx}=\frac{4\,\mathrm{Re}\{D\}-2\,\mathrm{Re}\{E\}}{c},\quad \sigma_{yy}=\frac{4\,\mathrm{Re}\{D\}+2\,\mathrm{Re}\{E\}}{c}$$

$$\sigma_{xy}=\frac{2\,\mathrm{Im}\{E\}}{c}=\frac{R_1^4\sigma_{xy}^\infty}{R_1^4+(\Gamma-1)(R_1^4-1)\left[\frac{1}{\kappa_1+1}-\sum_{j=1}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{1}{\kappa_j+1}-\frac{1}{\kappa_{j+1}+1}\right)\right]} \tag{25}$$

In addition the rigid-body rotation of the elliptical inhomogeneity can also be explicitly determined as

$$\varepsilon=\frac{(\kappa_1+1)\,\mathrm{Im}\{D\}}{c\mu_1}=\frac{R_1^2(1-\Gamma)\sigma_{xy}^\infty}{2\mu_1\Gamma\left[R_1^4+(\Gamma-1)(R_1^4-1)\left[\frac{1}{\kappa_1+1}-\sum_{j=1}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{1}{\kappa_j+1}-\frac{1}{\kappa_{j+1}+1}\right)\right]\right]} \tag{26}$$

which is always zero when $\Gamma=1$ or $\mu_1=\mu_N$.

The condition for the internal uniform stress field to be hydrostatic is $E=0$, or equivalently

$$\frac{\sigma_{yy}^\infty-\sigma_{xx}^\infty}{\sigma_{xx}^\infty+\sigma_{yy}^\infty}=R_1^{-2}(\kappa_N+1)\left[\frac{(1-\Gamma)(\kappa_1-1)}{(\kappa_1+1)(2+\Gamma\kappa_1-\Gamma)}+\sum_{j=1}^{N-1}\frac{R_1^2}{R_j^2}\left(\frac{1}{\kappa_j+1}-\frac{1}{\kappa_{j+1}+1}\right)\right],\quad \sigma_{xy}^\infty=0 \tag{27}$$

For $N=3$, the above condition reduces to

$$R_1^2\frac{\sigma_{yy}^\infty-\sigma_{xx}^\infty}{\sigma_{xx}^\infty+\sigma_{yy}^\infty}=\frac{(\kappa_3+1)(\kappa_2-1-\Gamma\kappa_1+\Gamma)+\rho(\kappa_3-\kappa_2)(2+\Gamma\kappa_1-\Gamma)}{(\kappa_2+1)(2+\Gamma\kappa_1-\Gamma)} \tag{28}$$

where $\rho=R_1^2/R_2^2$. Equation (28) just recovers Ru's result [4] for the case of $\beta_2=1$.

Now that we have determined the two constants D and E, as a result the two introduced analytic functions $\phi_0(\zeta)$ and $\psi_0(\zeta)$ have been completely obtained. The original analytic functions $\phi_m(\zeta)$ and $\psi_m(\zeta)$ $(m=2\sim N)$ defined in the interphases and in the matrix can then be determined from Eqs. (6) and (9).

4 Conclusions and Discussions

We derived the internal uniform stresses and rigid-body rotation for an N-phase elliptical inhomogeneity with $(N-1)$ confocal elliptical interfaces subjected to remote uniform in-plane stresses. In this research we only considered the case in which the interphases and the matrix possess the same shear modulus. In fact, the internal stress field is also uniform when the interphases and the inhomogeneity possess the same shear modulus. In this case the mean stress $(\sigma_{xx}+\sigma_{yy})$ is constant within each interphase. The conclusion of the uniform stress field inside the elliptical inhomogeneity is still valid in a more complex setting in which the inner portion of the interphases and the inhomogeneity possess the same shear modulus while the outer portion of the interphases and the matrix possess the same shear modulus. More specifically we assume that the $(M-1)$ inner interphases S_m $(m=2\sim M)$ and the inhomogeneity possess the same shear modulus, and that the rest $(N-M-1)$ outer interphases S_m $[m=(M+1)\sim(N-1)]$ and the surrounding matrix possess the same shear modulus. We found that even though the expressions of the internal stress components σ_{xx} and σ_{yy} are rather tedious, the internal shear stress and the rigid-body rotation can still be concisely determined as

$$\sigma_{xy}=\frac{R_M^4\sigma_{xy}^\infty}{R_M^4+(\Gamma-1)(R_M^4-1)\left[\frac{1}{\kappa_M+1}-\sum_{j=M}^{N-1}\frac{R_M^2}{R_j^2}\left(\frac{1}{\kappa_j+1}-\frac{1}{\kappa_{j+1}+1}\right)\right]} \tag{29}$$

$$\varepsilon=\frac{R_M^2(1-\Gamma)\sigma_{xy}^\infty}{2\mu_1\Gamma\left[R_M^4+(\Gamma-1)(R_M^4-1)\left[\frac{1}{\kappa_M+1}-\sum_{j=M}^{N-1}\frac{R_M^2}{R_j^2}\left(\frac{1}{\kappa_j+1}-\frac{1}{\kappa_{j+1}+1}\right)\right]\right]} \tag{30}$$

which indicate that the shear stress and rotation within the inhomogeneity are independent of Poisson's ratios ν_m $(m=1\sim M)$ of the inhomogeneity and the $(M-1)$ inner interphases.

In this work we discussed the layered inhomogeneity problem in the mapped ζ-plane. The inhomogeneity problems can be more conveniently discussed in the original z-plane when the whole composite possesses a constant shear modulus. As demonstrations, we consider in Appendix A an arbitrary number of arbitrary shaped inhomogeneities embedded in an infinite matrix and in Appendix B an N-phase inhomogeneity with $(N-1)$ interfaces of arbitrary shape. Furthermore the results in the main body and in Appendixes A and B can be further extended to finite plane strain deformations of compressible hyperelastic harmonic materials (see Appendix C).

Acknowledgment

Part of the work in this article was started while the author was at the University of Akron, supported by AFOSR Grant No. FA9550-06-1-0317. The author gratefully acknowledges inspiring and fruitful discussions with Professor Ernian Pan. The reviewers' comments are highly appreciated.

Appendix A: Interacting Arbitrary Shaped Inhomogeneities

Now we consider N arbitrary shaped inhomogeneities embedded in an unbounded matrix. Let S_i $(i=1-N)$ and S_m denote the N inhomogeneities and their surrounding matrix Γ_i $(i=1-N)$ the perfect interface separating S_i and S_m. All of the inhomogeneities and the matrix possess the same shear modulus. The matrix is subjected to remote uniform stresses. In the following the subscripts i $(i=1-N)$ and m are used to identify the associated quantities in S_i $(i=1-N)$ and S_m. The boundary conditions along the interfaces Γ_i $(i=1-N)$ can be expressed into

$$\begin{aligned}&\phi_i(z)+z\overline{\phi_i'(z)}+\overline{\psi_i(z)}=\phi_m(z)+z\overline{\phi_m'(z)}+\overline{\psi_m(z)},\\&\kappa_i\phi_i(z)-z\overline{\phi_i'(z)}-\overline{\psi_i(z)}=\kappa_m\phi_m(z)-z\overline{\phi_m'(z)}-\overline{\psi_m(z)},\\&z\in\Gamma_i\quad(i=1-N)\end{aligned} \tag{A1}$$

Adding together the two expressions in Eq. (A1), we obtain

$$\phi_m(z)=\frac{\kappa_i+1}{\kappa_m+1}\phi_i(z),\quad z\in\Gamma_i\quad(i=1-N) \tag{A2}$$

It is apparent from the above expression that

$$\phi_i(z)=\frac{\kappa_m+1}{\kappa_i+1}\Gamma z,\quad z\in S_i\quad(i=1-N)$$

$$\phi_m(z)=\Gamma z,\quad z\in S_m \tag{A3}$$

where $\Gamma=(\sigma_{xx}^\infty+\sigma_{yy}^\infty)/4$. Once we obtain $\phi_i(z)$ and $\phi_m(z)$, Eq. (A1) can be re-expressed into

$$\psi_m(z)=\psi_i(z)+2\Gamma\frac{\kappa_m-\kappa_i}{\kappa_i+1}D_i(z),\quad z\in\Gamma_i\quad(i=1-N) \tag{A4}$$

where $\bar{z}=D_i(z)$ along the interface Γ_i $(i=1-N)$. In addition $D_i(z)$ is analytic in the exterior of Γ_i except at infinity where it has a pole of finite degree $D_i(z)=P_i(z)+o(1)$, $|z|\to\infty$ [9]. It is more reasonable to write Eq. (A4) into the following form:

$$\begin{aligned}&\psi_m(z)-\Gamma'z-2\Gamma\frac{\kappa_m-\kappa_i}{\kappa_i+1}[D_i(z)-P_i(z)]\\&\quad=\psi_i(z)-\Gamma'z-2\Gamma\frac{\kappa_i-\kappa_m}{\kappa_i+1}P_i(z),\quad z\in\Gamma_i\quad(i=1-N)\end{aligned} \tag{A5}$$

where $\Gamma'=(\sigma_{yy}^\infty-\sigma_{xx}^\infty+2i\sigma_{xy}^\infty)/2$. In view of the above expression, we then introduce an analytic function $\psi_0(z)$ defined by

$$\psi_0(z) = \psi_i(z) - \Gamma' z - 2\Gamma\frac{\kappa_i - \kappa_m}{\kappa_i + 1}P_i(z) - 2\Gamma\sum_{j=1,j\neq i}^{N}\frac{\kappa_m - \kappa_j}{\kappa_j + 1}[D_j(z) - P_j(z)] \quad (i = 1 - N)$$

$$= \psi_m(z) - \Gamma' z - 2\Gamma\sum_{j=1}^{N}\frac{\kappa_m - \kappa_j}{\kappa_j + 1}[D_j(z) - P_j(z)] \qquad \text{(A6)}$$

Apparently the above introduced $\psi_0(z)$ is continuous across the interfaces Γ_i, $(i=1-N)$ and then analytic in the whole z-plane including the point at infinity. As a result we arrive at

$$\psi_i(z) = \Gamma' z + 2\Gamma\frac{\kappa_i - \kappa_m}{\kappa_i + 1}P_i(z) + 2\Gamma\sum_{j=1,j\neq i}^{N}\frac{\kappa_m - \kappa_j}{\kappa_j + 1}[D_j(z) - P_j(z)], \quad z \in S_i \quad (i = 1 - N)$$

$$\psi_m(z) = \Gamma' z + 2\Gamma\sum_{j=1}^{N}\frac{\kappa_m - \kappa_j}{\kappa_j + 1}[D_j(z) - P_j(z)], \quad z \in S_m \quad \text{(A7)}$$

In the following we consider several special cases. First we consider an isolated arbitrary shaped inhomogeneity S_1. In this case it follows from Eq. (A7) that

$$\psi_1(z) = \Gamma' z + 2\Gamma\frac{\kappa_1 - \kappa_m}{\kappa_1 + 1}P_1(z), \quad z \in S_1$$

$$\psi_m(z) = \Gamma' z + 2\Gamma\frac{\kappa_m - \kappa_1}{\kappa_1 + 1}[D_1(z) - P_1(z)], \quad z \in S_m \quad \text{(A8)}$$

We observe from Eqs. (A3) and (A8) that it is very simple to calculate the stress field inside an isolated inhomogeneity because the polynomial $P_1(z)$ can be explicitly determined [9]. It is observed from Eq. (A8) that the stress field inside the inhomogeneity is uniform only when the interface is an ellipse. When the inhomogeneity and the matrix have different shear moduli, we have to resort to the complicated Faber polynomials [5].

Second we consider the problem of two circular inhomogeneities embedded in an unbounded matrix. One circular inhomogeneity S_1 of radius R_1 is centered at $z=z_1$, and the other circular inhomogeneity S_2 of radius R_2 is centered at $z=z_2$. The analytic functions in the composite can be easily derived as

$$\phi_1(z) = \frac{\kappa_m + 1}{\kappa_1 + 1}\Gamma z, \quad \psi_1(z) = \Gamma' z + \frac{2(\kappa_m - \kappa_2)}{\kappa_2 + 1}\frac{\Gamma R_2^2}{z - z_2}, \quad z \in S_1$$

$$\phi_2(z) = \frac{\kappa_m + 1}{\kappa_2 + 1}\Gamma z, \quad \psi_2(z) = \Gamma' z + \frac{2(\kappa_m - \kappa_1)}{\kappa_1 + 1}\frac{\Gamma R_1^2}{z - z_1}, \quad z \in S_2$$

$$\phi_m(z) = \Gamma z, \quad \psi_m(z) = \Gamma' z + \frac{2(\kappa_m - \kappa_1)}{\kappa_1 + 1}\frac{\Gamma R_1^2}{z - z_1} + \frac{2(\kappa_m - \kappa_2)}{\kappa_2 + 1}\frac{\Gamma R_2^2}{z - z_2}, \quad z \in S_m \quad \text{(A9)}$$

which clearly indicate the nonuniform stress field inside each of the inhomogeneities due to the perturbation caused by the neighboring inhomogeneity.

If all the interfaces are *circular*, the loading is not confined to remote uniform stresses. In fact, when there is an arbitrary type singularity (e.g., an edge dislocation, a line force, a moment, or even an Eshelby inclusion of arbitrary shape [9,10]) located in the matrix, the singular behaviors of $\phi_m(z)$ and $\psi_m(z)$ are given by

$$\phi_m(z) = f_0(z), \quad \psi_m(z) = g_0(z) \qquad \text{(A10)}$$

As a result the analytic functions in the N circular inhomogeneities can be derived as

$$\phi_i(z) = \frac{\kappa_m + 1}{\kappa_i + 1}f_0(z)$$

$$\psi_i(z) = g_0(z) + \frac{\kappa_i - \kappa_m}{\kappa_i + 1}\frac{R_i^2[f_0'(z) - f_0'(z_i)]}{z - z_i} + \frac{\kappa_i - \kappa_m}{\kappa_i + 1}\bar{z}_i f_0'(z), + \sum_{j=1,j\neq i}^{N}\left[\frac{\kappa_m - \kappa_j}{\kappa_j + 1}\overline{f_0\left(\frac{R_j^2}{\bar{z} - \bar{z}_j} + z_j\right)} + \frac{\kappa_m - \kappa_j}{\kappa_j + 1}\frac{R_j^2 f_0'(z_j)}{z - z_j}\right] \quad z \in S_i \quad (i = 1 - N) \quad \text{(A11)}$$

and those in the matrix are

$$\phi_m(z) = f_0(z)$$

$$\psi_m(z) = g_0(z) + \sum_{j=1}^{N}\left[\frac{\kappa_m - \kappa_j}{\kappa_j + 1}\overline{f_0\left(\frac{R_j^2}{\bar{z} - \bar{z}_j} + z_j\right)} + \frac{\kappa_m - \kappa_j}{\kappa_j + 1}\frac{R_j^2 f_0'(z_j)}{z - z_j}\right], \quad z \in S_m \qquad \text{(A12)}$$

where R_i and z_i are the radius and center coordinate of the circular inhomogeneity S_i. It can be easily checked from Eqs. (A11) and (A12) that $\phi_i(z)$ and $\psi_i(z)$ are regular, whereas $\phi_m(z)$ and $\psi_m(z)$ exhibit the singular behaviors $f_0(z)$ and $g_0(z)$, respectively.

The above expressions (A11) and (A12) indicate that once the analytic functions $f_0(z)$ and $g_0(z)$ for a singularity in a homogeneous matrix material are known, the solutions to the problem of the singularity interacting with the N circular inhomogeneities can be written down immediately by using Eqs. (A11) and (A12). This result is remarkable and attractive even though the assumption of equal shear modulus has been made! As a check Eq. (A9) can also be obtained from Eqs. (A11) and (A12) by taking into mind that $f_0(z)=\Gamma z$ and $g_0(z)=\Gamma' z$.

Remark 1. In Hill's discussion [7], Hill stated that the problem can be fully solved when all the phases comprising the composite possess equal shear modulus but different Poisson's ratios. The original problem is reduced to the determination of the gravitational (or harmonic) potential ϕ. In order to obtain the gravitational potential ϕ, we have to carry out a volume integral $\int_V \mathrm{d}v/|\mathbf{r}-\mathbf{r}'|$, where V is the volume occupied by the misfitting inclusion with uniform volumetric eigenstrains [11]. If the shapes of the inhomogeneities are arbitrary and irregular, the volume integral is still difficult to obtain, especially for the points outside the inhomogeneities. On the other hand, when addressing two-dimensional (2D) Eshelby inclusions of arbitrary shape, however, the complex variable method provides a powerful tool [9]. The above analysis and the ensuing one in Appendix B demonstrate that the 2D inhomogeneity problem can be easily solved even the shape of the interface is arbitrary.

Appendix B: *N*-Phase Inhomogeneities With $(N-1)$ Interfaces of Arbitrary Shape

Similar to the discussion in the main body, here we consider an N-phase inhomogeneity with $(N-1)$ interfaces. Let S_1, S_m $[m=2\sim(N-1)]$, and S_N denote the inhomogeneity, the $(N-2)$ interphases, and the unbounded matrix, all of which are perfectly bonded through the $(N-1)$ interfaces Γ_m $[m=1\sim(N-1)]$. In the following discussion we will adopt the notation rule specified in Sec. 2. Now all the interfaces are of arbitrary shape, the interface Γ_m is always surrounded by the interface Γ_{m+1}, and all the N phases possess the same shear modulus. The matrix is subjected to remote uniform stresses. The boundary conditions along the interfaces Γ_m $[m=1\sim(N-1)]$ can be expressed into

$$\phi_m(z)+z\overline{\phi_m'(z)}+\overline{\psi_m(z)}=\phi_{m+1}(z)+z\overline{\phi_{m+1}'(z)}+\overline{\psi_{m+1}(z)},$$
$$\kappa_m\phi_m(z)-z\overline{\phi_m'(z)}-\overline{\psi_m(z)}=\kappa_{m+1}\phi_{m+1}(z)-z\overline{\phi_{m+1}'(z)}-\overline{\psi_{m+1}(z)},$$
$$z\in\Gamma_m\quad[m=1\sim(N-1)]\tag{B1}$$

Adding together the two expressions in Eq. (B1), we obtain

$$(\kappa_m+1)\phi_m(z)=(\kappa_{m+1}+1)\phi_{m+1}(z),\quad z\in\Gamma_m\quad[m=1\sim(N-1)]\tag{B2}$$

It is apparent from the above expression that

$$\phi_m(z)=\frac{\kappa_N+1}{\kappa_m+1}\Gamma z,\quad z\in S_m\ (m=1\sim N)\tag{B3}$$

where $\Gamma=(\sigma_{xx}^\infty+\sigma_{yy}^\infty)/4$. Once we obtain $\phi_m(z)$, Eq. (B1) can be re-expressed into

$$\psi_m(z)-2\Gamma\left(\frac{\kappa_N+1}{\kappa_{m+1}+1}-\frac{\kappa_N+1}{\kappa_m+1}\right)P_m(z)=\psi_{m+1}(z)-2\Gamma\left(\frac{\kappa_N+1}{\kappa_m+1}-\frac{\kappa_N+1}{\kappa_{m+1}+1}\right)[D_m(z)-P_m(z)]$$
$$z\in\Gamma_m\quad[m=1\sim(N-1)]\tag{B4}$$

where $\bar{z}=D_m(z)$ along the interface Γ_m $[m=1\sim(N-1)]$. In addition $D_m(z)$ is analytic in the exterior of Γ_m except at infinity where it has a pole of finite degree $D_m(z)=P_m(z)+o(1)$, $|z|\to\infty$. In view of the above expression, we then introduce an analytic function $\psi_0(z)$ defined by

$$\psi_0(z)=\psi_1(z)-\Gamma'z-2\Gamma\sum_{j=1}^{N-1}P_j(z)\left(\frac{\kappa_N+1}{\kappa_{j+1}+1}-\frac{\kappa_N+1}{\kappa_j+1}\right)=\psi_m(z)$$
$$-\Gamma'z-2\Gamma\sum_{j=m}^{N-1}P_j(z)\left(\frac{\kappa_N+1}{\kappa_{j+1}+1}-\frac{\kappa_N+1}{\kappa_j+1}\right)-2\Gamma\sum_{j=1}^{m-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)[D_j(z)-P_j(z)]\quad[m=2-(N-1)]$$
$$=\psi_N(z)-\Gamma'z-2\Gamma\sum_{j=1}^{N-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)[D_j(z)-P_j(z)]\tag{B5}$$

where $\Gamma'=(\sigma_{yy}^\infty-\sigma_{xx}^\infty+2i\sigma_{xy}^\infty)/2$. Apparently $\psi_0(z)$ is continuous across all the $(N-1)$ interfaces Γ_m $[m=1\sim(N-1)]$ and then analytic in the whole z-plane including the point at infinity. Thus $\psi_0(z)\equiv 0$. Consequently we arrive at

$$\psi_1(z)=\Gamma'z+2\Gamma\sum_{j=1}^{N-1}P_j(z)\left(\frac{\kappa_N+1}{\kappa_{j+1}+1}-\frac{\kappa_N+1}{\kappa_j+1}\right),\quad z\in S_1$$
$$\psi_m(z)=\Gamma'z+2\Gamma\sum_{j=1}^{N-1}P_j(z)\left(\frac{\kappa_N+1}{\kappa_{j+1}+1}-\frac{\kappa_N+1}{\kappa_j+1}\right)+2\Gamma\sum_{j=1}^{m-1}D_j(z)\times\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right),\quad z\in S_m\quad[m=2\sim(N-1)]$$
$$\psi_N(z)=\Gamma'z+2\Gamma\sum_{j=1}^{N-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)[D_j(z)-P_j(z)],\quad z\in S_N\tag{B6}$$

It is observed from the first expression of Eq. (B6) that the internal stresses are still uniform when all the interfaces are ellipses (here the elliptical interfaces are not confined to confocal and the centers of the ellipses are not necessarily common). More specifically here we assume that the semimajor and semiminor axes of the ellipse Γ_m are, respectively, a_m and b_m, and the major axis of the ellipse Γ_m is at an angle θ_m ($|\theta_m|\le\pi/2$) with the positive x-axis. In this case $\psi_1(z)$ can be simply determined as

$$\psi_1(z)=z\left[\Gamma'+2\Gamma\sum_{j=1}^{N-1}e^{-2i\theta_j}\frac{a_j-b_j}{a_j+b_j}\left(\frac{\kappa_N+1}{\kappa_{j+1}+1}-\frac{\kappa_N+1}{\kappa_j+1}\right)\right],\quad z\in S_1\tag{B7}$$

which clearly indicates that the internal stress field is uniform. Consequently the condition for the internal uniform stress field to be hydrostatic can be derived as

$$\frac{\sigma_{yy}^\infty-\sigma_{xx}^\infty}{\sigma_{xx}^\infty+\sigma_{yy}^\infty}=\sum_{j=1}^{N-1}\cos(2\theta_j)\frac{a_j-b_j}{a_j+b_j}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)$$
$$\frac{2\sigma_{xy}^\infty}{\sigma_{xx}^\infty+\sigma_{yy}^\infty}=\sum_{j=1}^{N-1}\sin(2\theta_j)\frac{a_j-b_j}{a_j+b_j}\left(\frac{\kappa_N+1}{\kappa_{j+1}+1}-\frac{\kappa_N+1}{\kappa_j+1}\right)\tag{B8}$$

which is consistent with Eq. (27) if all the interfaces are confocal. Furthermore it is also observed from the first expression of Eq. (B6) that the uniformity of the internal stresses for an inhomogeneity with nonelliptical interfaces is possible if the higher-order terms in $\psi_1(z)$ are canceled out.

If all the interfaces are *circular*, simple and closed-form solutions still exist for an arbitrary type singularity located in the matrix. The singular behaviors of $\phi_N(z)$ and $\psi_N(z)$ are given by

$$\phi_N(z)=f_0(z),\quad \psi_N(z)=g_0(z)\tag{B9}$$

Thus we have

$$\phi_m(z)=\frac{\kappa_N+1}{\kappa_m+1}f_0(z),\quad z\in S_m\quad(m=1\sim N).$$
$$\psi_1(z)=g_0(z)+\sum_{j=1}^{N-1}\left(\frac{\kappa_N+1}{\kappa_{j+1}+1}-\frac{\kappa_N+1}{\kappa_j+1}\right)\left[\bar{z}_jf_0'(z)+\frac{R_j^2[f_0'(z)-f_0'(z_j)]}{z-z_j}\right],\quad z\in S_1$$
$$\psi_m(z)=g_0(z)+\sum_{j=m}^{N-1}\left(\frac{\kappa_N+1}{\kappa_{j+1}+1}-\frac{\kappa_N+1}{\kappa_j+1}\right)\left[\bar{z}_jf_0'(z)+\frac{R_j^2[f_0'(z)-f_0'(z_j)]}{z-z_j}\right]+\sum_{j=1}^{m-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\times\left[\overline{f_0\left(\frac{R_j^2}{\bar{z}-\bar{z}_j}+z_j\right)}+\frac{R_j^2f_0'(z_j)}{z-z_j}\right],$$
$$z\in S_m,\ [m=2\sim(N-1)]\tag{B10}$$
$$\psi_N(z)=g_0(z)+\sum_{j=1}^{N-1}\left(\frac{\kappa_N+1}{\kappa_j+1}-\frac{\kappa_N+1}{\kappa_{j+1}+1}\right)\left[\overline{f_0\left(\frac{R_j^2}{\bar{z}-\bar{z}_j}+z_j\right)}+\frac{R_j^2f_0'(z_j)}{z-z_j}\right],\quad z\in S_N\tag{B11}$$

where R_m and z_m are the radius and center coordinate of the circle Γ_m.

Appendix C: Extension to Harmonic Materials

Interestingly the results in the main body and in Appendixes A and B can be simply extended to finite plane strain deformations of compressible hyperelastic harmonic materials [12–15]. Let the complex variable $z=x_1+ix_2$ be the initial coordinates of a material particle in the undeformed configuration and $w(z)=y_1(z)+iy_2(z)$ the corresponding spatial coordinates in the deformed configuration. Define the deformation gradient tensor as

$$F_{ij}=\frac{\partial y_i}{\partial x_j} \tag{C1}$$

For a particular class of harmonic materials, the strain energy density W defined with respect to the undeformed unit area can be expressed by

$$W=2\mu[F(I)-J], \quad F'(I)=\frac{1}{4\alpha}[I+\sqrt{I^2-16\alpha\beta}] \tag{C2}$$

Here I and J are the scalar invariants of FF^T given by

$$I=\lambda_1+\lambda_2=\sqrt{F_{ij}F_{ij}+2J}, \quad J=\lambda_1\lambda_2=\det[F] \tag{C3}$$

where λ_1 and λ_2 are the principal stretches, μ is the shear modulus, and $1/2\le\alpha<1$, $\beta>0$ are two material constants.

According to the formulation developed by Ru [14], the deformation $w(z)$ can be written in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ as

$$iw(z)=\alpha\varphi(z)+i\overline{\psi(z)}+\frac{\beta z}{\overline{\varphi'(z)}} \tag{C4}$$

and the complex Piola stress function $\chi(z)$ is given by

$$\chi(z)=2i\mu\left[(\alpha-1)\varphi(z)+i\overline{\psi(z)}+\frac{\beta z}{\overline{\varphi'(z)}}\right] \tag{C5}$$

In addition, the Piola stress components can be written in terms of the Piola stress function χ as

$$-\sigma_{21}+i\sigma_{11}=\chi_{,2}, \quad \sigma_{22}-i\sigma_{12}=\chi_{,1} \tag{C6}$$

In the following we endeavor to extend the results in the main body and in Appendixes A and B to hyperelastic harmonic materials.

C.1 *N*-Phase Elliptical Inhomogeneities With Internal Uniform Stresses in Finite Plane Elastostatics. As an extension of the results in the main body, we first address an N-phase elliptical inhomogeneity with internal uniform stresses in finite plane elastostatics. The problem discussed below is similar to that addressed in the main body except that now the composite is composed of harmonic materials. The $(N-2)$ interphases and the matrix possess the same shear modulus, and the formed $(N-1)$ interfaces are confocal.

The boundary conditions along the outer $(N-2)$ interfaces $|\zeta|=R_m$ $[m=2\sim(N-1)]$ can be expressed as

$$\alpha_m\varphi_m(\zeta)+i\overline{\psi_m(\zeta)}+\frac{\beta_m\overline{\omega'(\zeta)}\omega(\zeta)}{\overline{\varphi_m'(\zeta)}}=\alpha_{m+1}\varphi_{m+1}(\zeta)+i\overline{\psi_{m+1}(\zeta)}+\frac{\beta_{m+1}\overline{\omega'(\zeta)}\omega(\zeta)}{\overline{\varphi_{m+1}'(\zeta)}}$$

$$(\alpha_m-1)\varphi_m(\zeta)+i\overline{\psi_m(\zeta)}+\frac{\beta_m\overline{\omega'(\zeta)}\omega(\zeta)}{\overline{\varphi_m'(\zeta)}}=(\alpha_{m+1}-1)\varphi_{m+1}(\zeta)+i\overline{\psi_{m+1}(\zeta)}+\frac{\beta_{m+1}\overline{\omega'(\zeta)}\omega(\zeta)}{\overline{\varphi_{m+1}'(\zeta)}}, \quad |\zeta|=R_m \quad [m=2\sim(N-1)] \tag{C7}$$

Subtracting $(C7)_2$ from $(C7)_1$ yields

$$\varphi_m(\zeta)=\varphi_{m+1}(\zeta), \quad |\zeta|=R_m \quad [m=2\sim(N-1)] \tag{C8}$$

which implies that the introduced function $\varphi_0(\zeta)=\varphi_m(\zeta)$ $(m=2\sim N)$ is continuous across the interfaces $|\zeta|=R_m$ $[m=2\sim(N-1)]$ and then analytic within the region $|\zeta|>R_1$ except at infinity. The singular behavior of $\varphi_0(z)$ at infinity is

$$\varphi_0(\zeta)=\frac{cA}{2}\zeta+o(1), \quad |\zeta|\to\infty \tag{C9}$$

where the two complex constants A and B are related to the remote uniform Piola stresses σ_{11}^∞, σ_{22}^∞, σ_{12}^∞, and σ_{21}^∞ through the following relations:

$$-\sigma_{21}^\infty+i\sigma_{11}^\infty=2\mu_N\left[(1-\alpha_N)A+i\bar{B}-\frac{\beta_N}{\bar{A}}\right]$$

$$i\sigma_{22}^\infty+\sigma_{12}^\infty=2\mu_N\left[(1-\alpha_N)A-i\bar{B}-\frac{\beta_N}{\bar{A}}\right] \tag{C10}$$

Now that the interface conditions in Eq. (C7) can be reexpressed into

$$\psi_m(\zeta)+i(\alpha_m-\alpha_{m+1})\bar{\varphi}_0(R_m^2/\zeta)=\psi_{m+1}(\zeta)+i\frac{(\beta_{m+1}-\beta_m)\omega'(\zeta)\bar{\omega}(R_m^2/\zeta)}{\varphi_0'(\zeta)}$$

$$|\zeta|=R_m \quad [m=2\sim(N-1)] \tag{C11}$$

In view of Eq. (C11), we then introduce another analytic function $\psi_0(\zeta)$ defined by

$$\begin{aligned}\psi_0(\zeta)&=\psi_2(\zeta)+i\sum_{j=2}^{N-1}(\alpha_j-\alpha_{j+1})\bar{\varphi}_0(R_j^2/\zeta)\\&=\psi_m(\zeta)+i\sum_{j=m}^{N-1}(\alpha_j-\alpha_{j+1})\bar{\varphi}_0(R_j^2/\zeta)+i\sum_{j=2}^{m-1}\frac{(\beta_{j+1}-\beta_j)\omega'(\zeta)\bar{\omega}(R_j^2/\zeta)}{\varphi_0'(\zeta)}\\&\qquad[m=3\sim(N-1)]\\&=\psi_N(\zeta)+i\sum_{j=2}^{N-1}\frac{(\beta_{j+1}-\beta_j)\omega'(\zeta)\bar{\omega}(R_j^2/\zeta)}{\varphi_0'(\zeta)}\end{aligned} \tag{C12}$$

It can be easily observed from the above expression that $\psi_0(\zeta)$ is also continuous across the interfaces $|\zeta|=R_m$ $[m=2\sim(N-1)]$ and then analytic within the region $|\zeta|>R_1$ except at infinity. The singular behavior of $\psi_0(\zeta)$ at infinity is

$$\psi_0(\zeta)=\left[\frac{cB}{2}+\frac{ci}{2A}\sum_{j=2}^{N-1}R_j^{-2}(\beta_{j+1}-\beta_j)\right]\zeta+o(1), \quad |\zeta|\to\infty \tag{C13}$$

To ensure the uniform Piola stress field inside the elliptical inhomogeneity, we have to assume that $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ must take the following forms:

$$\varphi_1(\zeta)=D(\zeta+\zeta^{-1}), \quad \psi_1(\zeta)=E(\zeta+\zeta^{-1}) \tag{C14}$$

where D and E and two complex constants to be determined. The boundary conditions along the inner interface $|\zeta|=R_1$ can be expressed into

$$\alpha_2\varphi_2(\zeta)+i\overline{\psi_2(\zeta)}+\frac{\beta_2\overline{\omega'(\zeta)}\omega(\zeta)}{\overline{\varphi_2'(\zeta)}}=\left(\alpha_1D+i\bar{E}R_1^{-2}+\frac{\beta_1c^2}{4\bar{D}}\right)\zeta+\left(\alpha_1D+i\bar{E}R_1^2+\frac{\beta_1c^2}{4\bar{D}}\right)\zeta^{-1}$$

$$(\alpha_2-1)\varphi_2(\zeta)+i\overline{\psi_2(\zeta)}+\frac{\beta_2\overline{\omega'(\zeta)}\omega(\zeta)}{\overline{\varphi_2'(\zeta)}} = \Gamma\left((\alpha_1-1)D+i\bar{E}R_1^{-2}+\frac{\beta_1c^2}{4\bar{D}}\right)\zeta + \Gamma\left((\alpha_1-1)D+i\bar{E}R_1^{2}+\frac{\beta_1c^2}{4\bar{D}}\right)\zeta^{-1}, \quad |\zeta|=R_1 \tag{C15}$$

where $\Gamma=\mu_1/\mu_N$. The above can be equivalently expressed in terms of $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ as

$$\alpha_2\varphi_0(\zeta)+\sum_{j=2}^{N-1}(\alpha_{j+1}-\alpha_j)\varphi_0\left(\frac{R_j^2}{R_1^2}\zeta\right)-\frac{cA}{2}\left[\alpha_2+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}(\alpha_{j+1}-\alpha_j)\right]\zeta+\left[\frac{icR_1^2\bar{B}}{2}+\frac{c\beta_2}{2\bar{A}}+\frac{cR_1^2}{2\bar{A}}\sum_{j=2}^{N-1}R_j^{-2}(\beta_{j+1}-\beta_j)-\left(\alpha_1D+i\bar{E}R_1^2+\frac{\beta_1c^2}{4\bar{D}}\right)\right]\zeta^{-1} = -i\bar{\psi}_0(R_1^2/\zeta)-\frac{\beta_2\bar{\omega}'(R_1^2/\zeta)\omega(\zeta)}{\bar{\varphi}_0'(R_1^2/\zeta)} + \left[\alpha_1D+i\bar{E}R_1^{-2}+\frac{\beta_1c^2}{4\bar{D}}-\frac{cA}{2}\left[\alpha_2+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}(\alpha_{j+1}-\alpha_j)\right]\right]\zeta + \left[\frac{icR_1^2\bar{B}}{2}+\frac{c\beta_2}{2\bar{A}}+\frac{cR_1^2}{2\bar{A}}\sum_{j=2}^{N-1}R_j^{-2}(\beta_{j+1}-\beta_j)\right]\zeta^{-1}, \quad |\zeta|=R_1 \tag{C16}$$

$$(\alpha_2-1)\varphi_0(\zeta)+\sum_{j=2}^{N-1}(\alpha_{j+1}-\alpha_j)\varphi_0\left(\frac{R_j^2}{R_1^2}\zeta\right)-\frac{cA}{2}\left[(\alpha_2-1)+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}(\alpha_{j+1}-\alpha_j)\right]\zeta+\left[\frac{icR_1^2\bar{B}}{2}+\frac{c\beta_2}{2\bar{A}}+\frac{cR_1^2}{2\bar{A}}\sum_{j=2}^{N-1}R_j^{-2}(\beta_{j+1}-\beta_j)-\Gamma\left((\alpha_1-1)D+i\bar{E}R_1^2+\frac{\beta_1c^2}{4\bar{D}}\right)\right]\zeta^{-1} = -i\bar{\psi}_0(R_1^2/\zeta)-\frac{\beta_2\bar{\omega}'(R_1^2/\zeta)\omega(\zeta)}{\bar{\varphi}_0'(R_1^2/\zeta)}+\left[\Gamma\left((\alpha_1-1)D+i\bar{E}R_1^{-2}+\frac{\beta_1c^2}{4\bar{D}}\right)-\frac{cA}{2}\left[(\alpha_2-1)+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}(\alpha_{j+1}-\alpha_j)\right]\right]\zeta+\left[\frac{icR_1^2\bar{B}}{2}+\frac{c\beta_2}{2\bar{A}}+\frac{cR_1^2}{2\bar{A}}\sum_{j=2}^{N-1}R_j^{-2}(\beta_{j+1}-\beta_j)\right]\zeta^{-1}, \quad |\zeta|=R_1 \tag{C17}$$

Apparently the left-hand side of Eq. (C16) and that of Eq. (C17) are analytic outside the circle $|\zeta|=R_1$ including the point at infinity, while the right-hand side of Eqs. (C16) and that of Eq. (C17) are analytic within the circle $|\zeta|=R_1$. Consequently the two sides of both Eqs. (C16) and (C17) should be identically zero such that

$$\alpha_2\varphi_0(\zeta)+\sum_{j=2}^{N-1}(\alpha_{j+1}-\alpha_j)\varphi_0\left(\frac{R_j^2}{R_1^2}\zeta\right)=\frac{cA}{2}\left[\alpha_2+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}(\alpha_{j+1}-\alpha_j)\right]\zeta-\left[\frac{icR_1^2\bar{B}}{2}+\frac{c\beta_2}{2\bar{A}}+\frac{c}{2\bar{A}}\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}(\beta_{j+1}-\beta_j)-\left(\alpha_1D+i\bar{E}R_1^2+\frac{\beta_1c^2}{4\bar{D}}\right)\right]\zeta^{-1} \tag{C18}$$

$$i\bar{\psi}_0(R_1^2/\zeta)+\frac{\beta_2\bar{\omega}'(R_1^2/\zeta)\omega(\zeta)}{\bar{\varphi}_0'(R_1^2/\zeta)} = \left[\alpha_1D+i\bar{E}R_1^{-2}+\frac{\beta_1c^2}{4\bar{D}}-\frac{cA}{2}\left[\alpha_2+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}(\alpha_{j+1}-\alpha_j)\right]\right]\zeta + \left[\frac{icR_1^2\bar{B}}{2}+\frac{c\beta_2}{2\bar{A}}+\frac{c}{2\bar{A}}\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}(\beta_{j+1}-\beta_j)\right]\zeta^{-1} \tag{C19}$$

$$(\alpha_2-1)\varphi_0(\zeta)+\sum_{j=2}^{N-1}(\alpha_{j+1}-\alpha_j)\varphi_0\left(\frac{R_j^2}{R_1^2}\zeta\right)=\frac{cA}{2}\left[(\alpha_2-1)+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}(\alpha_{j+1}-\alpha_j)\right]\zeta-\left[\frac{icR_1^2\bar{B}}{2}+\frac{c\beta_2}{2\bar{A}}+\frac{c}{2\bar{A}}\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}(\beta_{j+1}-\beta_j)-\Gamma\left((\alpha_1-1)D+i\bar{E}R_1^2+\frac{\beta_1c^2}{4\bar{D}}\right)\right]\zeta^{-1} \tag{C20}$$

$$i\bar{\psi}_0(R_1^2/\zeta)+\frac{\beta_2\bar{\omega}'(R_1^2/\zeta)\omega(\zeta)}{\bar{\varphi}_0'(R_1^2/\zeta)} = \left[\Gamma\left((\alpha_1-1)D+i\bar{E}R_1^{-2}+\frac{\beta_1c^2}{4\bar{D}}\right)-\frac{cA}{2}\left[(\alpha_2-1)+\sum_{j=2}^{N-1}\frac{R_j^2}{R_1^2}(\alpha_{j+1}-\alpha_j)\right]\right]\zeta+\left[\frac{icR_1^2\bar{B}}{2}+\frac{c\beta_2}{2\bar{A}}+\frac{c}{2\bar{A}}\sum_{j=2}^{N-1}\frac{R_1^2}{R_j^2}(\beta_{j+1}-\beta_j)\right]\zeta^{-1} \tag{C21}$$

The solutions of $\varphi_0(\zeta)$ can be easily obtained from Eqs. (C18) and (C20) as

$$\varphi_0(\zeta)=\frac{cA}{2}\zeta-\frac{\dfrac{icR_1^2\bar{B}}{2}+\dfrac{c\beta_2}{2\bar{A}}+\dfrac{c}{2\bar{A}}\sum_{j=2}^{N-1}\dfrac{R_1^2}{R_j^2}(\beta_{j+1}-\beta_j)-\left(\alpha_1D+i\bar{E}R_1^2+\dfrac{\beta_1c^2}{4\bar{D}}\right)}{\alpha_2+\sum_{j=2}^{N-1}\dfrac{R_1^2}{R_j^2}(\alpha_{j+1}-\alpha_j)}\zeta^{-1} \tag{C22}$$

$$\varphi_0(\zeta)=\frac{cA}{2}\zeta-\frac{\dfrac{icR_1^2\bar{B}}{2}+\dfrac{c\beta_2}{2\bar{A}}+\dfrac{c}{2\bar{A}}\sum_{j=2}^{N-1}\dfrac{R_1^2}{R_j^2}(\beta_{j+1}-\beta_j)-\Gamma\left((\alpha_1-1)D+i\bar{E}R_1^2+\dfrac{\beta_1c^2}{4\bar{D}}\right)}{\alpha_2-1+\sum_{j=2}^{N-1}\dfrac{R_1^2}{R_j^2}(\alpha_{j+1}-\alpha_j)}\zeta^{-1} \tag{C23}$$

The compatibility condition of $\varphi_0(\zeta)$ and that of $i\bar{\psi}_0(R_1^2/\zeta)+\beta_2\bar{\omega}'(R_1^2/\zeta)\omega(\zeta)/\bar{\varphi}_0'(R_1^2/\zeta)$ will yield

$$\frac{\dfrac{icR_1^2\bar{B}}{2}+\dfrac{c\beta_2}{2\bar{A}}+\dfrac{c}{2\bar{A}}\sum_{j=2}^{N-1}\dfrac{R_1^2}{R_j^2}(\beta_{j+1}-\beta_j)-\left(\alpha_1 D+i\bar{E}R_1^2+\dfrac{\beta_1 c^2}{4\bar{D}}\right)}{\alpha_2+\sum_{j=2}^{N-1}\dfrac{R_1^2}{R_j^2}(\alpha_{j+1}-\alpha_j)}=\frac{\dfrac{icR_1^2\bar{B}}{2}+\dfrac{c\beta_2}{2\bar{A}}+\dfrac{c}{2\bar{A}}\sum_{j=2}^{N-1}\dfrac{R_1^2}{R_j^2}(\beta_{j+1}-\beta_j)-\Gamma\left((\alpha_1-1)D+i\bar{E}R_1^2+\dfrac{\beta_1 c^2}{4\bar{D}}\right)}{\alpha_2-1+\sum_{j=2}^{N-1}\dfrac{R_1^2}{R_j^2}(\alpha_{j+1}-\alpha_j)} \tag{C24}$$

$$i\bar{E}=\frac{(\Gamma-1)\alpha_1-\Gamma}{1-\Gamma}R_1^2 D-\frac{\beta_1 R_1^2 c^2}{4\bar{D}}+\frac{cR_1^2 A}{2(1-\Gamma)} \tag{C25}$$

The two unknowns D and E can then be determined by solving the above two nonlinear equations. The above analysis demonstrates that the Piola stresses within the N-phase elliptical inhomogeneity are uniform. Interestingly the existence of the internal uniform hydrostatic Piola stresses ($\sigma_{11}=\sigma_{22}$, $\sigma_{12}=\sigma_{21}=0$) can be further discussed.

Remark 2. Similar to the discussion in Ref. [15], the following condition must be satisfied:

$$\varphi_0'(\zeta)\neq 0,\quad |\zeta|>1 \tag{C26}$$

or equivalently

$$A\neq-\frac{iR_1^2\bar{B}+\dfrac{\beta_2}{\bar{A}}+\dfrac{1}{\bar{A}}\sum_{j=2}^{N-1}\dfrac{R_1^2}{R_j^2}(\beta_{j+1}-\beta_j)-\dfrac{2}{c}\left(\alpha_1 D+i\bar{E}R_1^2+\dfrac{\beta_1 c^2}{4\bar{D}}\right)}{\alpha_2+\sum_{j=2}^{N-1}\dfrac{R_1^2}{R_j^2}(\alpha_{j+1}-\alpha_j)}\zeta^{-2},\quad |\zeta|>1 \tag{C27}$$

C.2 *N* Interacting Arbitrary Shaped Inhomogeneities in Finite Plane Elastostatics. As an extension of the results in Appendix A, we next address N arbitrary shaped inhomogeneities embedded in an unbounded matrix within the framework of finite deformation of harmonic materials. Let S_j ($j=1-N$) and S_m denote the N inhomogeneities and their surrounding matrix, Γ_j ($j=1-N$) the perfect interface separating S_j and S_m. All of the inhomogeneities and the matrix possess the same shear modulus. In addition the matrix is subjected to remote uniform Piola stresses σ_{11}^∞, σ_{22}^∞, σ_{12}^∞, and σ_{21}^∞. The boundary conditions along the interfaces Γ_j ($j=1-N$) can be expressed into

$$\alpha_j\varphi_j(z)+i\overline{\psi_j(z)}+\frac{\beta_j z}{\overline{\varphi_j'(z)}}=\alpha_m\varphi_m(z)+i\overline{\psi_m(z)}+\frac{\beta_m z}{\overline{\varphi_m'(z)}},$$

$$(\alpha_j-1)\varphi_j(z)+i\overline{\psi_j(z)}+\frac{\beta_j z}{\overline{\varphi_j'(z)}}=(\alpha_m-1)\varphi_m(z)+i\overline{\psi_m(z)}+\frac{\beta_m z}{\overline{\varphi_m'(z)}},$$

$$z\in\Gamma_j\quad(j=1-N) \tag{C28}$$

Subtracting $(C28)_2$ from $(C28)_1$ yields

$$\varphi_j(z)=\varphi_m(z),\quad z\in\Gamma_j\quad(j=1-N) \tag{C29}$$

which implies that

$$\varphi_m(z)=\varphi_j(z)=Az\quad(j=1-N) \tag{C30}$$

where the two complex constants A and B are related to the remote uniform Piola stresses σ_{11}^∞, σ_{22}^∞, σ_{12}^∞, and σ_{21}^∞ through the following relations:

$$-\sigma_{21}^\infty+i\sigma_{11}^\infty=2\mu_m\left[(1-\alpha_m)A+i\bar{B}-\frac{\beta_m}{\bar{A}}\right]$$

$$i\sigma_{22}^\infty+\sigma_{12}^\infty=2\mu_m\left[(1-\alpha_m)A-i\bar{B}-\frac{\beta_m}{\bar{A}}\right] \tag{C31}$$

Once we obtain $\varphi_j(z)$ and $\varphi_m(z)$, Eq. (C28) can be re-expressed into

$$\psi_j(z)-i\left[(\alpha_m-\alpha_j)\bar{A}+\frac{\beta_m-\beta_j}{A}\right]P_j(z)=\psi_m(z)-i\left[(\alpha_j-\alpha_m)\bar{A}+\frac{\beta_j-\beta_m}{A}\right][D_j(z)-P_j(z)],\quad z\in\Gamma_j\quad(j=1-N) \tag{C32}$$

Similar to the discussion in Appendix A, it follows from the above expression that

$$\psi_j(z)=Bz+i\left[(\alpha_m-\alpha_j)\bar{A}+\frac{\beta_m-\beta_j}{A}\right]P_j(z)+i\sum_{k=1,k\neq j}^{N}\left[(\alpha_k-\alpha_m)\bar{A}+\frac{\beta_k-\beta_m}{A}\right][D_k(z)-P_k(z)],\quad z\in S_j\quad(j=1-N)$$

$$\psi_m(z)=Bz+i\sum_{k=1}^{N}\left[(\alpha_k-\alpha_m)\bar{A}+\frac{\beta_k-\beta_m}{A}\right][D_k(z)-P_k(z)],\quad z\in S_m \tag{C33}$$

C.3 *N*-Phase Inhomogeneities With $(N-1)$ Interfaces of Arbitrary Shape in Finite Plane Elastostatics. As an extension of the results in Appendix B, we finally consider an N-phase inhomogeneity with $(N-1)$ interfaces within the framework of finite deformation of harmonic materials. All the interfaces are of arbitrary shape, and all the N phases possess the same shear modulus. The boundary conditions along the interfaces Γ_m [$m=1\sim(N-1)$] separating phases S_m and S_{m+1} can be expressed into

$$\alpha_m\varphi_m(z)+i\overline{\psi_m(z)}+\frac{\beta_m z}{\overline{\varphi_m'(z)}}=\alpha_{m+1}\varphi_{m+1}(z)+i\overline{\psi_{m+1}(z)}+\frac{\beta_{m+1} z}{\overline{\varphi_{m+1}'(z)}}$$

$$(\alpha_m - 1)\phi_m(z) + i\overline{\psi_m(z)} + \frac{\beta_m z}{\overline{\varphi_m'(z)}}$$

$$= (\alpha_{m+1} - 1)\varphi_{m+1}(z) + i\overline{\psi_{m+1}(z)} + \frac{\beta_{m+1} z}{\overline{\varphi_{m+1}'(z)}},$$

$$z \in \Gamma_m \quad [m = 1 \sim (N-1)] \tag{C34}$$

Subtracting $(C34)_2$ from $(C34)_1$ and applying Liouville's theorem, we finally obtain

$$\varphi_m(z) = Az, \quad z \in S_m \quad (m = 1 \sim N) \tag{C35}$$

where A and B have been determined by Eq. (C10).

Now that Eq. (C34) can be re-expressed into

$$\psi_m(z) - i\left[(\alpha_{m+1} - \alpha_m)\bar{A} + \frac{\beta_{m+1} - \beta_m}{A}\right] P_m(z)$$

$$= \psi_{m+1}(z) - i\left[(\alpha_m - \alpha_{m+1})\bar{A} + \frac{\beta_m - \beta_{m+1}}{A}\right][D_m(z) - P_m(z)],$$

$$z \in \Gamma_m \quad [m = 1 \sim (N-1)] \tag{C36}$$

Similar to the discussion in Appendix B, we can finally arrive at

$$\psi_1(z) = Bz + i\sum_{j=1}^{N-1} P_j(z)\left[(\alpha_{j+1} - \alpha_j)\bar{A} + \frac{\beta_{j+1} - \beta_j}{A}\right], \quad z \in S_1$$

$$\psi_m(z) = Bz + i\sum_{j=1}^{N-1} P_j(z)\left[(\alpha_{j+1} - \alpha_j)\bar{A} + \frac{\beta_{j+1} - \beta_j}{A}\right]$$

$$+ i\sum_{j=1}^{m-1} D_j(z)\left[(\alpha_j - \alpha_{j+1})\bar{A} + \frac{\beta_j - \beta_{j+1}}{A}\right],$$

$$z \in S_m \quad [m = 2 \sim (N-1)]$$

$$\psi_N(z) = Bz + i\sum_{j=1}^{N-1}\left[(\alpha_j - \alpha_{j+1})\bar{A} + \frac{\beta_j - \beta_{j+1}}{A}\right][D_j(z) - P_j(z)],$$

$$z \in S_N \tag{C37}$$

Apparently it is observed from the first expression of Eq. (C37) that (i) the internal Piola stresses are still uniform when all the interfaces are ellipses (here the elliptical interfaces are not confined to confocal and the centers of the ellipses are not necessarily common), and (ii) the uniformity of the internal Piola stresses for an inhomogeneity with nonelliptical interfaces is possible if the higher-order terms in $\psi_1(z)$ are canceled out. The above conclusion is similar to that drawn in Appendix B.

References

[1] Ru, C. Q., Schiavone, P., and Mioduchowski, A., 1998, "Uniformity of the Stresses Within a Three-Phase Elliptical Inclusion in Anti-Plane Shear," J. Elast., **52**, pp. 121–128.

[2] Christensen, R. M., and Lo, K. H., 1979, "Solutions for Effective Shear Properties in Three-Phase Sphere and Cylindrical Models," J. Mech. Phys. Solids, **27**, pp. 315–330.

[3] Luo, H. A., and Weng, G. J., 1989, "On Eshelby's *S*-Tensor in a Three-Phase Cylindrically Concentric Solid and the Elastic Moduli of Fiber-Reinforced Composites," Mech. Mater., **8**, pp. 77–88.

[4] Ru, C. Q., 1999, "Three-Phase Elliptical Inclusions With Internal Uniform Hydrostatic Stresses," J. Mech. Phys. Solids, **47**, pp. 259–273.

[5] Tsukrov, I., and Novak, J., 2004, "Effective Elastic Properties of Solids With Two-Dimensional Inclusions of Irregular Shape," Int. J. Solids Struct., **41**, pp. 6905–6924.

[6] Muskhelishvili, N. I., 1953, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen.

[7] Hill, R., 1963, "Elastic Properties of Reinforced Solids: Some Theoretical Principles," J. Mech. Phys. Solids, **11**, pp. 357–372.

[8] Hao, T. H., 2006, "Exact Solution of Plane Isolated Crack Normal to a Bimaterial Interface of Infinite Extent," Acta Mech. Sin., **22**, pp. 455–468.

[9] Ru, C. Q., 1999, "Analytic Solution for Eshelby's Problem of an Inclusion of Arbitrary Shape in a Plane or Half-Plane," ASME J. Appl. Mech., **66**, pp. 315–322.

[10] Zhigang, S., 1989, "Singularities Interacting With Interfaces and Cracks," Int. J. Solids Struct., **25**, pp. 1133–1142.

[11] Eshelby, J. D., 1957, "The Determination of the Elastic Field of an Ellipsoidal Inclusion, and Related Problems," Proc. R. Soc. London, Ser. A, **241**, pp. 376–396.

[12] Knowles, J. K., and Sternberg, E., 1975, "On the Singularity Induced by Certain Mixed Boundary Conditions in Linearized and Nonlinear Elastostatics," Int. J. Solids Struct., **11**, pp. 1173–1201.

[13] Varley, E., and Cumberbatch, E., 1980, "Finite Deformation of Elastic Materials Surrounding Cylindrical Hole," J. Elast., **10**, pp. 341–405.

[14] Ru, C. Q., 2002, "On Complex-Variable Formulation for Finite Plane Elastostatics of Harmonic Materials," Acta Mech., **156**, pp. 219–234.

[15] Ru, C. Q., Schiavone, P., Sudak, L. J., and Mioduchowski, A., 2005, "Uniformity of Stresses Inside an Elliptical Inclusion in Finite Elastostatics," Int. J. Nonlinear Mech., **40**, pp. 281–287.